Department of Aerospace Engineering

IIT Kanpur, India

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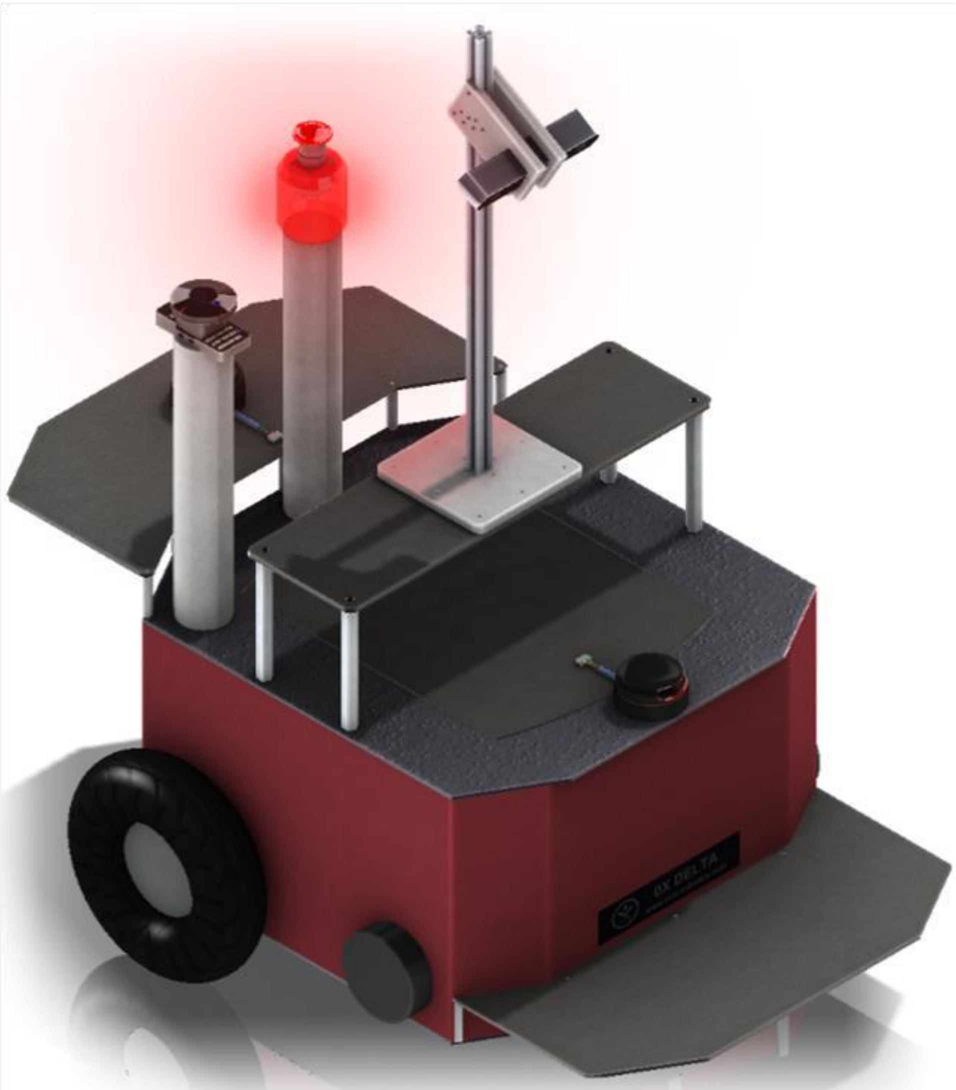
Autonomous Navigation of a Ground Vehicle



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Observability of a Linear System

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- System

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$z(t) = Cx(t)$$

- Output and its derivatives

$$z(t) = Cx(t),$$
$$\dot{z}(t) = C\dot{x}(t) = CAx(t) + CBu(t),$$
$$\ddot{z}(t) = CA^2x(t) + CABu(t) + CB\dot{u}(t).$$

- We can infer

$$Cx(t),\ CAx(t),\ CA^2x(t), \cdots,\ CA^{n-1}x(t).$$

- Known quantities

$$CBu(t), CABu(t), \text{and } CB\dot{(u)}(t)$$

$$O_{lin} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

$$rank(O_{lin}) = n$$

Observability of nonlinear system

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- The system is time invariant
- The system is linear in the control.
- A Lie derivative is the derivative of a scalar along integral curves of the vector field f

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{n} u_i(t) f_i(x(t))$$

$$z(t) = h(x(t))$$

$$L^0 h = h$$

$$L^1_{f_i} h = \nabla L^0 h \cdot f_i$$

$$L^2_{f_i f_i} h = \nabla L^1_{f_i} h \cdot f_i$$

$$L^2_{f_i f_j} h = \nabla L^1_{f_i} h \cdot f_j$$

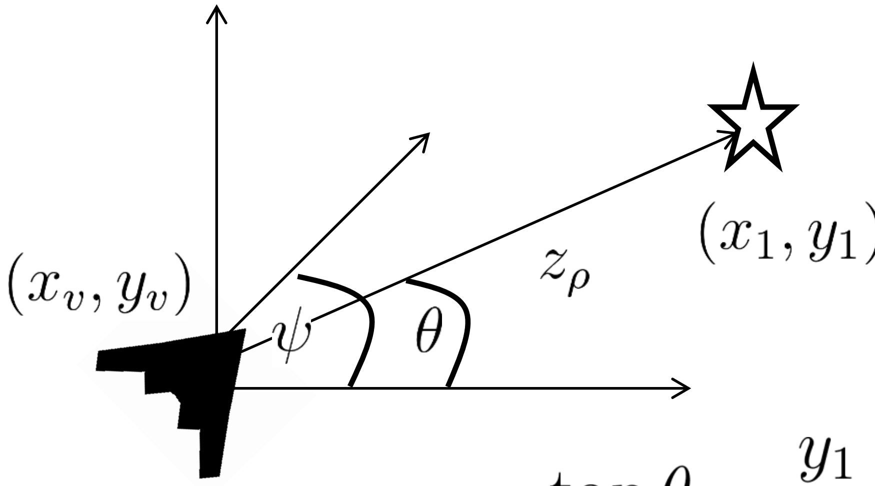
Credit: Dr. Rajnikant Sharma

$$O = \begin{pmatrix} \nabla L^0 h \\ \nabla L^1_{f_i} h \\ \nabla L^2_{f_i f_j} h \\ \vdots \end{pmatrix}$$

Example 1

$$\begin{pmatrix} \dot{x}_v \\ \dot{y}_v \end{pmatrix} = V \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

$$z_\rho = (x_1 - x_v)^2 + (y_1 - y_v)^2$$



$$\tan\theta = \frac{y_1 - y_v}{x_1 - x_v}$$

Lie derivatives

$$L^0 h = (x_1 - x_v)^2 + (y_1 - y_v)^2$$

$$\nabla L^0 h = \begin{bmatrix} -2(x_1 - x_v) & -2(y_1 - y_v) \end{bmatrix}$$

$$\frac{\dot{z}_p}{V} = L^1 h f_1 = \nabla L^0 h f_1 = -2((x_1 - x_v)\cos\psi + (y_1 - y_v)\sin\psi)$$

Not observable if

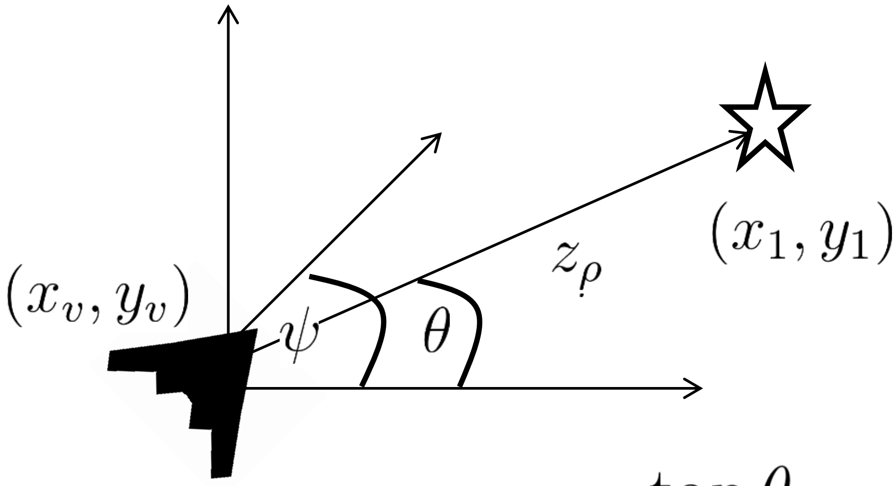
$$((x_1 - x_v)\cos\psi + (y_1 - y_v)\sin\psi) = 0 \rightarrow V \perp z_\rho$$

$$z_\rho = (x_1 - x_v)^2 + (y_1 - y_v)^2 = c$$

Example 2

$$\begin{pmatrix} \dot{x}_v \\ \dot{y}_v \end{pmatrix} = V \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

$$z = \tan^{-1}\frac{y_1 - y_v}{x_1 - x_v} - \psi$$



$$\tan\theta = \frac{y_1 - y_v}{x_1 - x_v}$$

Lie derivatives

$$L^0 h = z$$

$$\nabla L^0 h = \begin{bmatrix} -\frac{(y_1 - y_v)}{\rho^2} & \frac{(x_1 - x_v)}{\rho^2} \end{bmatrix}$$

$$\frac{\dot{z}}{V} = L^1 h f_1 = \nabla L^0 h f_1 = \frac{(-(y_1 - y_v)\cos\psi + (x_1 - x_v)\sin\psi)}{\rho^2}$$

Not observable if

$$(-(y_1 - y_v)\cos\psi + (x_1 - x_v)\sin\psi) = 0 \rightarrow \psi = \theta$$

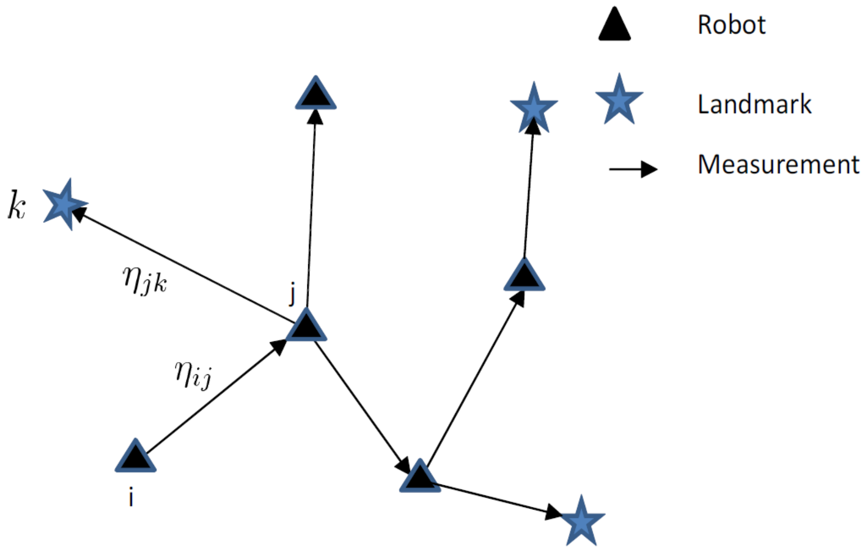
Bearing-only cooperative localization

- **Equation of Motion**

$$\dot{X}_i = f_i(X_i, u_i) \triangleq \begin{pmatrix} V_i \cos\psi_i \\ V_i \sin\psi_i \\ \omega_i \end{pmatrix}$$

- **Bearing Measurement**

$$\eta_{ij} = \tan^{-1}\left(\frac{y_j - y_i}{x_j - x_i}\right) - \psi_i.$$



$$G_n^l \triangleq \{\mathcal{V}_{n,l}, \mathcal{E}_{n,l}\}$$

Bearing-only cooperative localization

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- Joint states

$$X = [X_1^\top X_2^\top \cdots X_n^\top]^\top$$

- System

$$\Sigma : \begin{array}{c} \dot{X} = f(X,u) = [f_1^\top(X_1,u_1),\cdots,f_n^\top(X_n,u_n)]^\top \\ Y = h(X,Xl) = [h_1^\top(X,Xl)\cdots h_m^\top(X,Xl)]^\top \end{array}$$

- Can be implemented in a centralized or a decentralized manner using EKF(Roumeliotis2002), MMSE (Sanderson1998), MLE (Howard2002), Particle Filter (Fox2000), and MAP (Nerurkar2009)

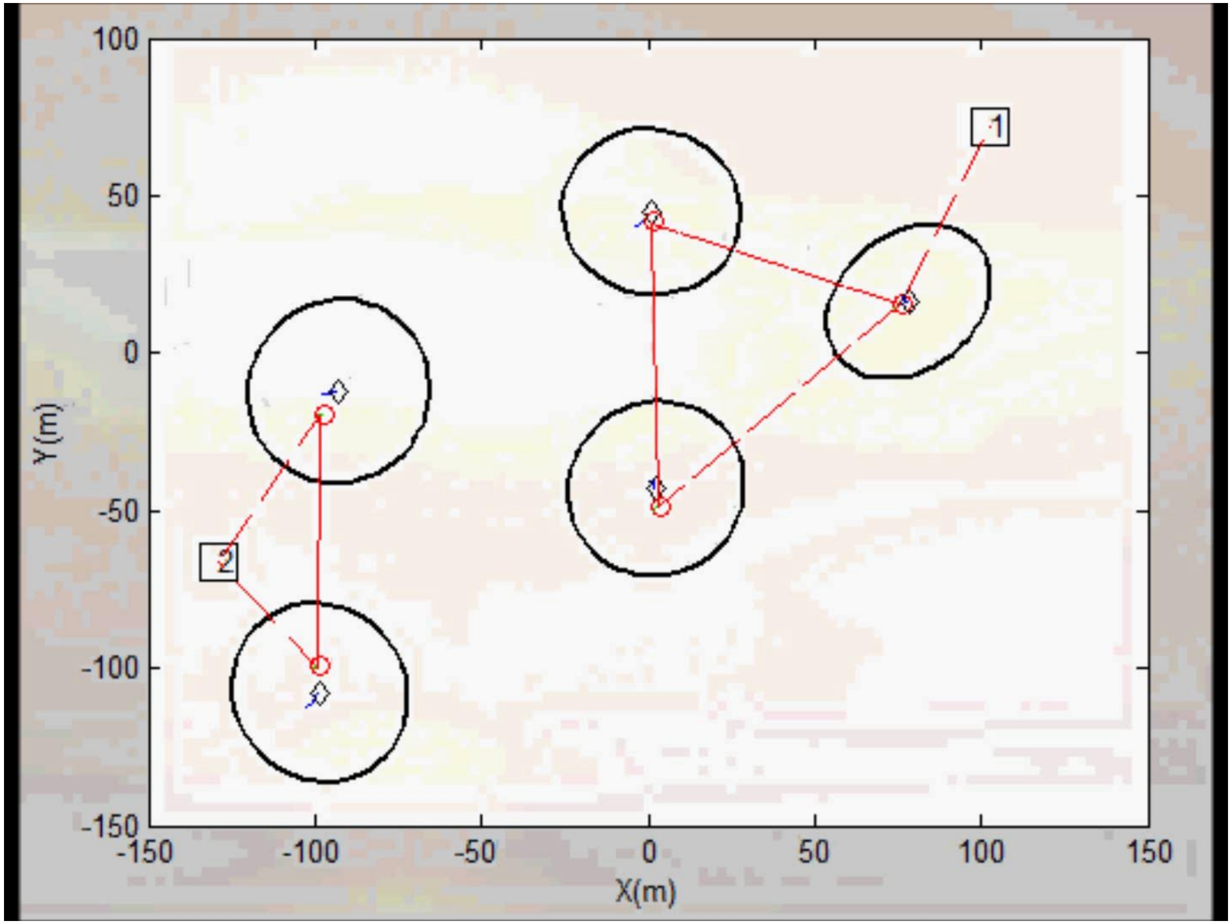
Graph-based Observability Analysis

- Roumeliotis2002, Bicchi1998, Huang2008, and Martinelli2005
- What effects the observability of the system?

* Control strategy

$$u = [u_1, \cdots, u_n]^\top \in R^{2n}$$

* Topology of sensor network (RPMG)
* Number of landmarks.



Edge between two vehicle nodes

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\dot{X} = f_{v_i} v_i + f_{\omega_i} \omega_i + f_{v_j} v_j + f_{\omega_j} \omega_j$$

$$y = h(X) = \eta_{ij}$$

$$O_{ij} = \begin{bmatrix} \nabla L^0 h \\ \nabla L^1_{f_{v_i}} h \\ \nabla L^1_{f_{v_j}} h \\ \nabla L^1_{f_{\omega_i}} h \\ \nabla L^2_{f_{v_i} f_{\omega_i}} h \\ \nabla L^2_{f_{v_j} f_{\omega_j}} h \end{bmatrix} = \begin{bmatrix} -\Delta y_{ij} & \Delta x_{ij} & -R^2_{ij} & \Delta y_{ij} & -\Delta x_{ij} & 0 \\ s\psi_i & -c\psi_i & J^+_i & -s\psi_i & c\psi_i & 0 \\ -s\psi_j & c\psi_j & 0 & s\psi_j & -c\psi_j & -J^+_j \\ -2\Delta x_{ij} & -2\Delta y_{ij} & 0 & 2\Delta x_{ij} & 2\Delta y_{ij} & 0 \\ 0 & 0 & -J_\psi & 0 & 0 & J_\psi \\ c\psi_i & s\psi_i & J^-_i & -c\psi_i & -s\psi_i & 0 \\ -c\psi_j & -s\psi_j & 0 & c\psi_j & s\psi_j & -J^-_j \end{bmatrix}.$$

Observability Analysis

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$$f_{v_i} = [c_{\psi_i}\ s_{\psi_i}\ 0\ 0\ 0\ 0]^\top,\ f_{\omega_i} = [0\ 0\ 1\ 0\ 0\ 0]^\top,\ f_{v_j} = [0\ 0\ 0\ c_{\psi_j}\ s_{\psi_j}\ 0]^\top,\ f_{\omega_j} = [0\ 0\ 0\ 0\ 0\ 1]^\top$$

$$L^0 h = \eta_{ij}$$

$$\nabla L^0 h = \left[\begin{array}{cccccc} -y_{ij} & x_{ij} & -R_{ij}^2 & y_{ij} & -x_{ij} & 0 \end{array}\right]$$

$$L^1_{f_{v_i}} h = \nabla L^0 h \cdot f_{v_i} = x_{ij} s_{\psi_i} - y_{ij} c_{\psi_i},$$

$$L^1_{f_{v_j}} h = \nabla L^0 h \cdot f_{v_j} = -(x_{ij} s_{\psi_j} - y_{ij} c_{\psi_j}),$$

$$L^1_{f_{\omega_i}} h = \nabla L^0 h \cdot f_{\omega_i} = -R_{ij}^2,$$

$$L^1_{f_{\omega_j}} h = \nabla L^0 h \cdot f_{\omega_j} = 0,$$

Observability Analysis

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$$\nabla L^1_{f_{v_i}} h = \begin{bmatrix} s_{\psi_i} & -c_{\psi_i} & J_i^+ & -s_{\psi_i} & c_{\psi_i} & 0 \end{bmatrix},$$

$$\nabla L^1_{f_{v_j}} h = \begin{bmatrix} -s_{\psi_j} & c_{\psi_j} & 0 & s_{\psi_j} & -c_{\psi_j} & -J_j^+ \end{bmatrix},$$

$$\nabla L^1_{f_{\omega_i}} h = 2\begin{bmatrix} -x_{ij} & -y_{ij} & 0 & x_{ij} & y_{ij} & 0 \end{bmatrix},$$

$$L^2_{f_{v_i} f_{v_i}} h = \nabla L^1_{f_{v_i}} h \cdot f_{v_i} = s_{\psi_i} c_{\psi_i} - s_{\psi_i} c_{\psi_i} = 0,$$

$$L^2_{f_{v_j} f_{v_j}} h = \nabla L^1_{f_{v_j}} h \cdot f_{v_j} = s_{\psi_j} c_{\psi_j} - s_{\psi_j} c_{\psi_j} = 0,$$

$$L^2_{f_{v_i} f_{v_j}} h = \nabla L^1_{f_{v_i}} h \cdot f_{v_j} = -s_{\psi_i} c_{\psi_j} + s_{\psi_j} c_{\psi_i},$$

$$L^2_{f_{v_i} f_{\omega_i}} h = \nabla L^1_{f_{v_i}} h \cdot f_{\omega_i} = J_i^+,$$

$$L^2_{f_{v_j} f_{\omega_j}} h = \nabla L^1_{f_{v_j}} h \cdot f_{\omega_j} = -J_j^+,$$

$$L^2_{f_{\omega_i} f_{v_i}} h = \nabla L^1_{f_{\omega_i}} h \cdot f_{v_i} = -2J_i^+,$$

$$L^2_{f_{\omega_i} f_{v_j}} h = \nabla L^1_{f_{\omega_i}} h \cdot f_{v_j} = 2J_j^+,$$

Observability Analysis

$$\nabla L^2_{f_{v_i} f_{v_j}} h = \begin{bmatrix} 0 & 0 & -J_\psi & 0 & 0 & J_\psi \end{bmatrix},$$

$$\nabla L^2_{f_{v_i} f_{\omega_i}} h = \begin{bmatrix} c_{\psi_i} & s_{\psi_i} & J_i^- & c_{\psi_i} & s_{\psi_i} & 0 \end{bmatrix},$$

$$\nabla L^2_{f_{v_j} f_{\omega_j}} h = \begin{bmatrix} -c_{\psi_j} & -s_{\psi_j} & 0 & s_{\psi_j} & -c_{\psi_j} & -J_j^- \end{bmatrix},$$

$$L^3_{f_{v_i} f_{v_j} f_{\omega_i}} h = \nabla L^2_{f_{v_i} f_{v_j}} h \cdot f_{\omega_i} = -(c_{\psi_i} c_{\psi_j} + s_{\psi_i} s_{\psi_j}),$$

$$L^3_{f_{v_i} f_{v_j} f_{\omega_j}} h = \nabla L^2_{f_{v_i} f_{v_j}} h \cdot f_{\omega_j} = (c_{\psi_i} c_{\psi_j} + s_{\psi_i} s_{\psi_j}),$$

$$L^3_{f_{v_i} f_{\omega_i} f_{v_i}} h = \nabla L^2_{f_{v_i} f_{\omega_i}} h \cdot f_{v_i} = 1,$$

$$L^3_{f_{v_j} f_{\omega_j} f_{v_j}} h = \nabla L^2_{f_{v_i} f_{\omega_j}} h \cdot f_{v_j} = 1,$$

$$L^3_{f_{v_i} f_{\omega_i} f_{\omega_i}} h = \nabla L^2_{f_{v_i} f_{\omega_i}} h \cdot f_{\omega_i} = -(x_{ij} s_{\psi_i} - y_{ij} c_{\psi_i}),$$

$$L^3_{f_{v_j} f_{\omega_j} f_{\omega_j}} h = \nabla L^2_{f_{v_i} f_{\omega_j}} h \cdot f_{\omega_j} = x_{ij} s_{\psi_j} - y_{ij} c_{\psi_j},$$

Observability Analysis

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$$\nabla L^3_{f_{v_i} f_{v_j} f_{\omega_i}} h = a_1 \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} = -\frac{a_1}{a_2} \nabla L^2_{f_{v_i} f_{v_j}} h,$$

$$\nabla L^3_{f_{v_i} f_{v_j} f_{\omega_j}} h = \frac{a_1}{a_2} \nabla L^2_{f_{v_i} f_{v_j}} h,$$

$$\nabla L^3_{f_{v_i} f_{\omega_i} f_{\omega_i}} h = -(x_{ij} s_{\psi_i} - y_{ij} c_{\psi_i}) = -\nabla L^1_{f_{v_i}} h,$$

$$\nabla L^3_{f_{v_j} f_{\omega_j} f_{\omega_j}} h = x_{ij} s_{\psi_j} - y_{ij} c_{\psi_j} = -\nabla L^1_{f_{v_j}} h,$$

Linearly dependent upon the previous gradients, therefore it does not contribute in rank of the observability matrix

Edge between two vehicle nodes

- **Lemma 1**
  - **Rank of the observability matrix O_{ij} is three if**
    - (1) $V_i > 0$
    - (2) $V_j > 0$
    - (3) $J_i^- = y_{ij}\cos\psi_i - x_{ij}\sin\psi_i \neq 0$
    - (4) $J_j^+ = x_{ij}\cos\psi_j + y_{ij}\sin\psi_j \neq 0$
- **Proof**

$$E_{ij}\mathcal{O}_{ij} = U_{ij} = \begin{bmatrix} \mathbf{I}_3 & \bar{O}_{ij} \\ \mathbf{0}_{4\times3} & \mathbf{0}_{4\times3} \end{bmatrix}$$

$$\bar{O}^{ij} = \begin{bmatrix} -1 & 0 & \Delta y_{ij} \\ 0 & -1 & \Delta x_{ij} \\ 0 & 0 & -1 \end{bmatrix}.$$

$$E_{ij} = \begin{bmatrix} -\frac{c_{\psi_j}J_i^+}{J_i^-J_j^+} & -\frac{c_{\psi_j}R_{ij}^2}{J_i^-J_j^+} & -\frac{y_{ij}}{J_j^+} & 0 & 0 & 0 & 0 \\ -\frac{s_{\psi_j}J_i^+}{J_i^-J_j^+} & -\frac{s_{\psi_j}R_{ij}^2}{J_i^-J_j^+} & \frac{x_{ij}}{J_j^+} & 0 & 0 & 0 & 0 \\ -\frac{s(\psi_j-\psi_i)}{J_i^-J_j^+} & \frac{J_j^-}{J_i^-J_j^+} & \frac{1}{J_j^+} & 0 & 0 & 0 & 0 \\ -\frac{2J_i^+}{J_i^-} & -\frac{2R_{ij}^2}{J_i^-} & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2}\frac{s(2\psi_i-2\psi_j)}{J_i^-J_j^+} & \frac{c(\psi_j-\psi_i)J_j^-}{J_i^-J_j^+} & \frac{c(\psi_i-\psi_j)}{J_j^+} & 0 & 1 & 0 & 0 \\ \frac{1}{J_i^-} & \frac{J_i^+}{J_i^-} & 0 & 0 & 0 & 1 & 0 \\ -\frac{J_i^+}{J_i^-J_j^+} & -\frac{R_{ij}^2}{J_i^-J_j^+} & -\frac{J_j^-}{J_j^+} & 0 & 0 & 0 & 1 \end{bmatrix},$$

Edge between vehicle and a landmark

$$X = X_1$$

$$\dot{X} = f_{v_i} v_i + f_{\omega_i} \omega_i$$

$$y = h(X_i, Xl_k) = \eta_{ik}$$

$$O_{ik} = \begin{bmatrix} -y_{ik} & x_{ik} & -R_{ik}^2 \\ s\psi_i & -c\psi_i & J^+ \\ -2x_{ik} & -2y_{ik} & 0 \\ c\psi_i & s\psi_i & J^- \end{bmatrix}$$

Edge between vehicle and a landmark

- Lemma 2
  - Rank of the observability matrix O_{ik} is two if
    - (1) $V_i > 0$
    - (2) $J^- = y_{ik}\cos\psi_i - y_{ik}\sin\psi_i \neq 0$
- Proof

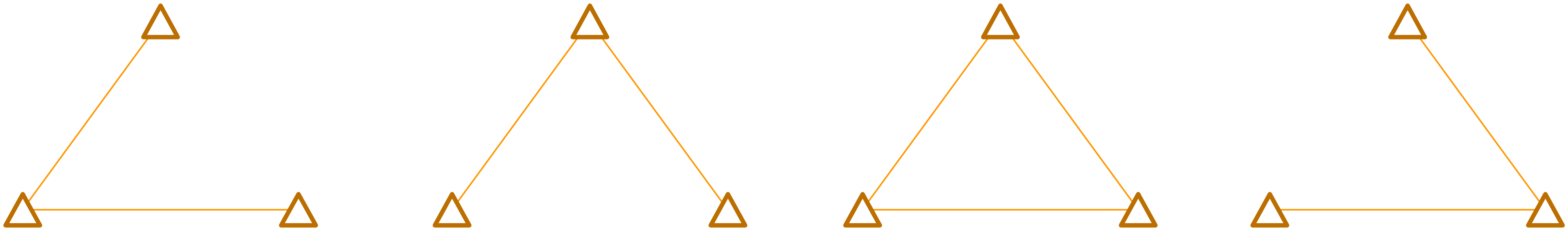
$$E_{ik}O_{ik} = U_{ik} = \begin{bmatrix} \bar{O}_{ik} \\ \mathbf{0}_{2\times 3} \end{bmatrix}$$

$$\bar{O}_{ik} = \begin{bmatrix} 1 & 0 & \Delta y_{ik} \\ 0 & 1 & -\Delta x_{ik} \end{bmatrix}.$$

$$E_{ik} = \begin{bmatrix} \frac{-c\psi_i}{J^-} & \frac{-x_{ik}}{J^-} & 0 & 0 \\ \frac{-s\psi_i}{J^-} & \frac{-y_{ik}}{J^-} & 0 & 0 \\ \frac{-2J^+}{J^-} & \frac{-2R_{ik}^2}{J^-} & 1 & 0 \\ \frac{1}{J^-} & \frac{J^+}{J^-} & 0 & 1 \end{bmatrix}$$

Three nodes

- **Lemma 3**
  - Observability matrices of all of the configurations of G_3^0 span the same space



$$O_a = \begin{bmatrix} \mathbf{I}_3 & \bar{O}_{12} & \mathbf{0} \\ \mathbf{I}_3 & \mathbf{0} & \bar{O}_{13} \end{bmatrix} \quad O_b = \begin{bmatrix} \mathbf{I}_3 & \bar{O}_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & \bar{O}_{23} \end{bmatrix} E_aO_a = E_bO_b = E_dO_d = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \bar{O}_{13} \\ \mathbf{0} & \mathbf{I}_3 & \bar{O}_{23} \end{bmatrix},$$

$$O_c = \begin{bmatrix} \mathbf{I}_3 & \bar{O}_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 & \bar{O}_{23} \\ \mathbf{I}_3 & \mathbf{0} & \bar{O}_{13} \end{bmatrix} \quad O_d = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \bar{O}_{13} \\ \mathbf{0} & \mathbf{I}_3 & \bar{O}_{23} \end{bmatrix}. \qquad E_cO_c = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} & \bar{O}_{13} \\ \mathbf{0} & \mathbf{I}_3 & \bar{O}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Three nodes

- Lemma 3
  - Observability matrices of all of the configurations of G_2^1 span the same space

- Proof

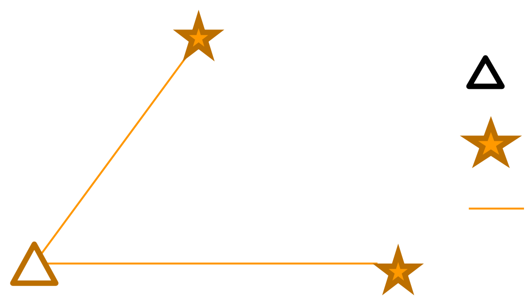
$$O_a^p = \left[\begin{array}{cc} \mathbf{I}_3 & \bar{O}_{ij} \\ \bar{O}l_{ik} & 0_{2\times 3} \end{array} \right], \; O_b^p = \left[\begin{array}{cc} \mathbf{I}_3 & \bar{O}_{ik} \\ 0_{2\times 3} & \bar{O}l_{jk} \end{array} \right].$$

$$E_{ijk} = \left[\begin{array}{cc} \mathbf{I}_3 & \mathbf{0}_{3\times 2} \\ -\bar{O}l_{ik} & \mathbf{I}_2 \end{array} \right] \qquad E_{ijk} O_a^p = O_b^p$$

Three nodes

- **Lemma 4**
  - **Position and heading of a robot is completely observable in RPMG**
- **Proof:**

$$G_1^2$$

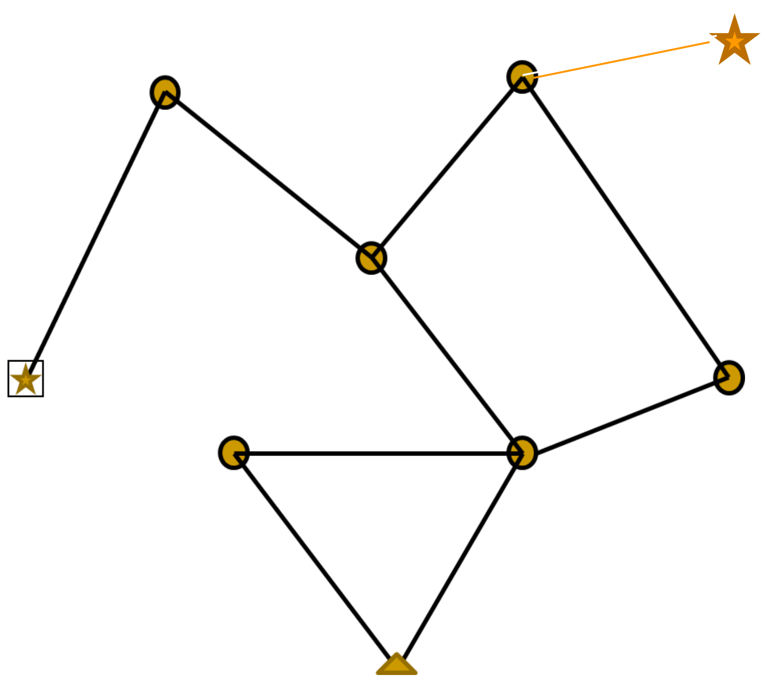


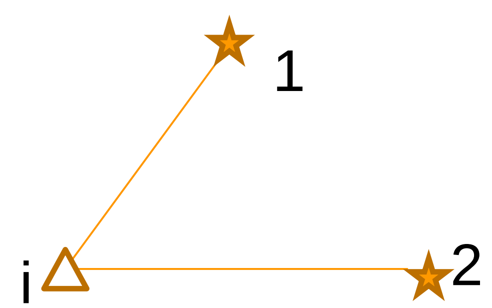
$$O_{i12}^{p} = \begin{bmatrix} \bar{O}l_{i1} \\ \bar{O}l_{i2} \end{bmatrix}$$

$$E_{i12} O_{i12}^{p} = \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{0}_{1\times 3} \end{bmatrix} \quad E_{i12} = \begin{bmatrix} \frac{y_{i2}}{y_{12}} & 0 & -\frac{y_{i1}}{y_{12}} & 0 \\ -\frac{x_{i1}}{y_{12}} & 1 & \frac{x_{i1}}{y_{12}} & 0 \\ -\frac{1}{y_{12}} & 0 & \frac{1}{y_{12}} & 0 \\ -\frac{x_{12}}{y_{12}} & -1 & -\frac{x_{12}}{y_{12}} & 1 \end{bmatrix}$$

General n-nodes

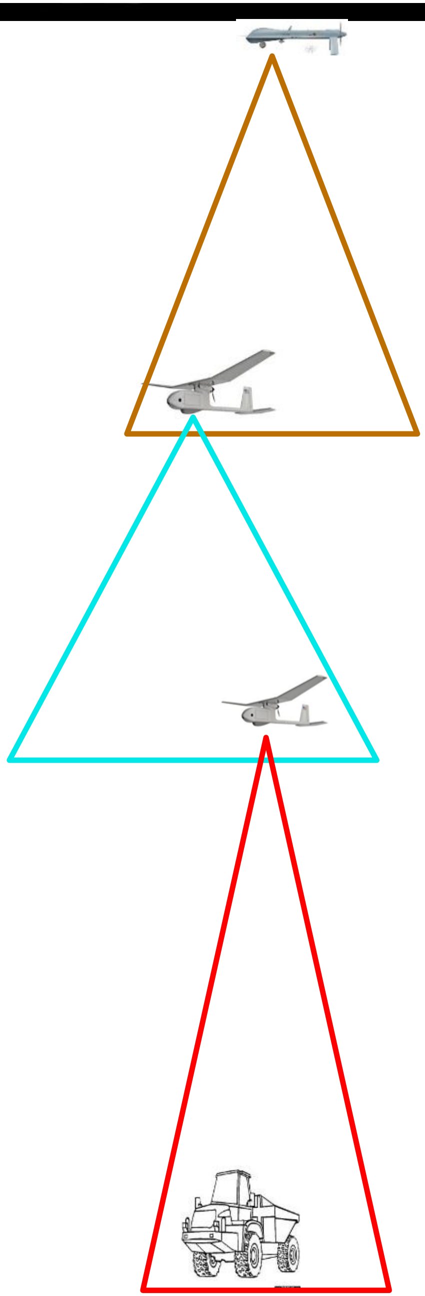
- **Theorem**
  - **If an RPMG is proper and each vehicle node has a path to two known landmarks then the system is completely observable.**





Cooperative geolocation

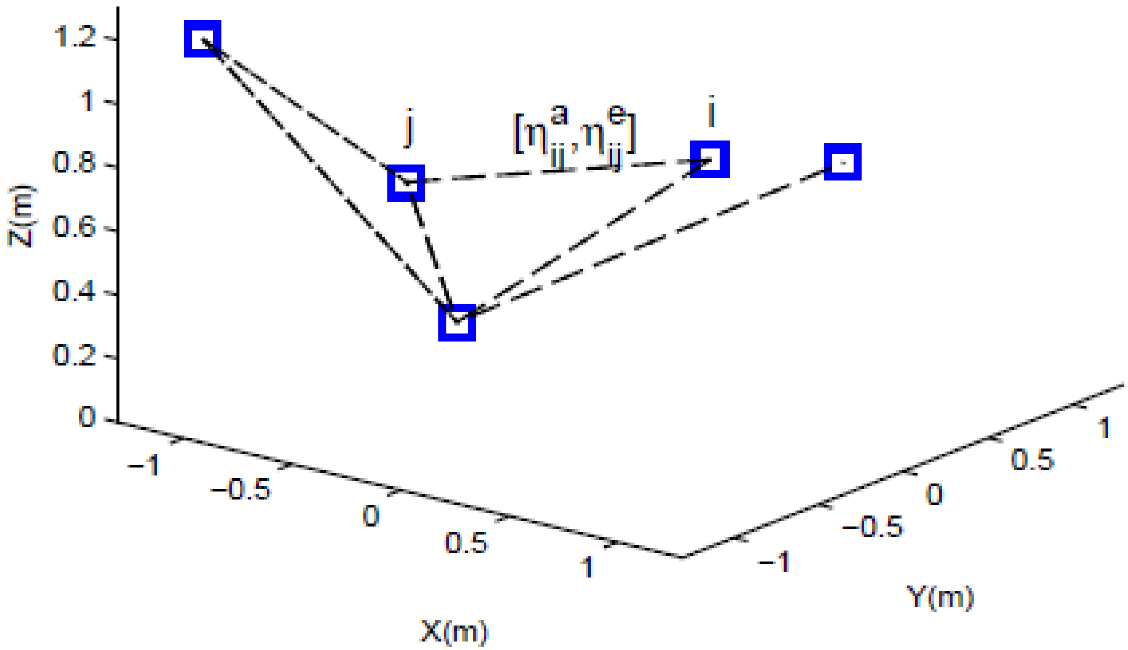
- **Exchange**
  - **Inter vehicle bearing measurement**
  - **Position and heading**
  - **Motion information (velocity, angular rates)**
  - **GPS only to An**
- **To Cooperatively estimate states (position and heading) of all of the vehicles.**
- **Is this system observable?**



Observability Result

- **Relative Position measurement graph(RPMG)**

$$G_n \triangleq \{\mathcal{V}_n, \mathcal{E}_n\}$$



$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \\ \dot{\psi}_i \end{pmatrix} = \begin{pmatrix} V_i \cos\theta_i \cos\psi_i \\ V_i \cos\theta_i \sin\psi_i \\ -V_i \sin\theta_i \\ \omega_i \end{pmatrix}$$

$$\omega_i = \frac{g}{V_i} \tan\phi_i$$

$$\eta^a_{ij} = \tan^{-1}\left(\frac{y_j - y_i}{x_j - x_i}\right) - \psi_i$$

$$\eta^e_{ij} = \tan^{-1}\left(\frac{z_j - z_i}{R_{ij}}\right) - \theta_i$$

Observability Result

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- Theorem
- The system is completely observable if
  - (1) the RPMG is proper and connected
  - (2) $V_i > 0, \quad i \in \mathcal{V}_n$
  - (3) $\theta_i \neq \frac{\pi}{2}, \quad i \in \mathcal{V}_n$
  - (4) one of the vehicle has GPS